Fourier Series.

Consider

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad (1)$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx \qquad (2)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \qquad (3)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx. \qquad (4)$$

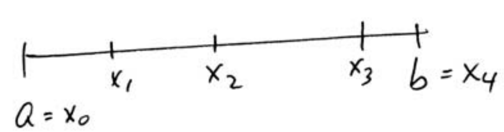
Does the infinite series (1) with coefficients (2)-(4) Converges?

Is this Series Converging to $f(x)$ for every x?

what Conditions should be imposed in $f(x)$?

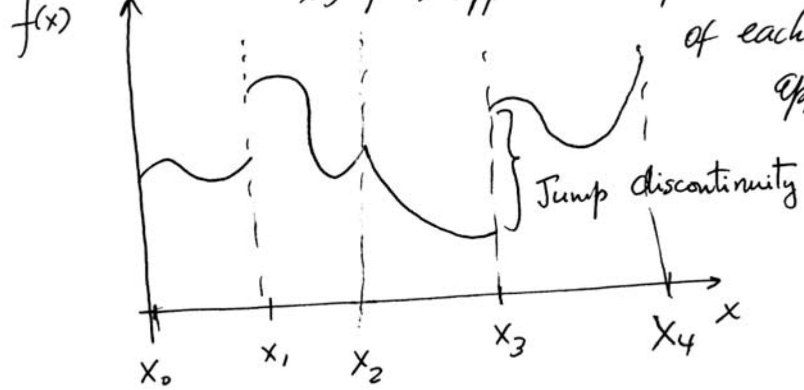
<u>Def.</u>- A function $f(x)$ is piecewise Smooth on $[a,b]$

If for a partition of $[a,b]$: $a = x_0 < x_1 < \dots x_n = b$



i) $f(x)$ and $f'(x)$ continuous on each open subinterval

ii) $f(x)$ approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.



Show others!

- Fourier series is periodic with period $2L$, since each function in the series is periodic with period $2L$.

3.2 <u>statement of convergence.</u>

<u>Def.</u> - The infinite series (1) with coefficients defined by (2)-(4) is called the Fourier series of $f(x)$ over the interval $-L \le x \le L$.

<u>Remarks:</u> a) (1) may not converge

b) (1) may converge to something different than $f(x)$..

The definition of the coefficients of the F.S. of $f(x)$ requires that

$$\left| \underbrace{\int_{-L}^{L} f(x)\,dx}_{= (2L)a_0} \right| < \infty.$$

Therefore, $f(x) = \frac{1}{x^2}$ does not have a F.S. on $[-L, L]$.

Since F.S. of $f(x)$ and $f(x)$ are in general two different functions. The following notation is adopted

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x.$$

Convergence Theorem for F.S.

If $f(x)$ is piecewise smooth on the interval $[-L, L]$ then the series of $f(x)$ converges

1.- to the periodic extension of $f(x)$, where the periodic extension is continuous.

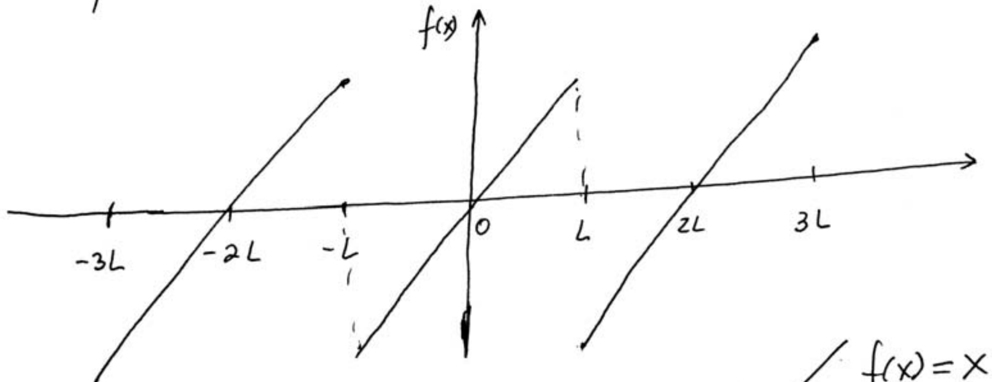
2.- To the average of the two limits

$$\frac{1}{2}\left[f(x^+) + f(x^-) \right]$$

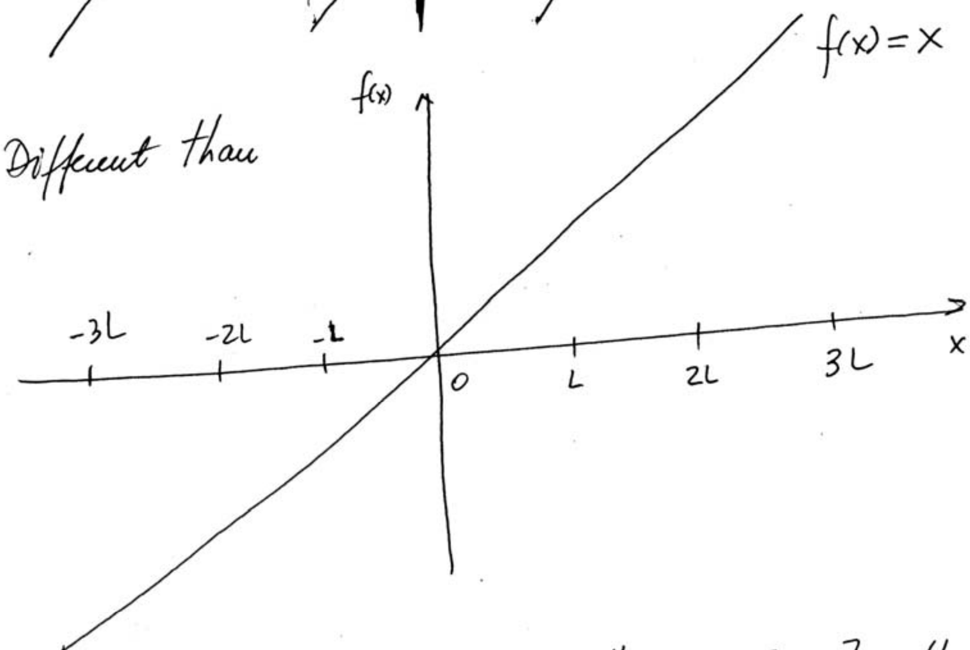
where the periodic extension has a jump.

Example: $f(x) = x, \quad -L \le x \le L.$

periodic extension is



Different than



If $f(x)$ is piecewise smooth on $[-L, L]$, then

$$\frac{f(x^+) + f(x^-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

for $-L < x < L$

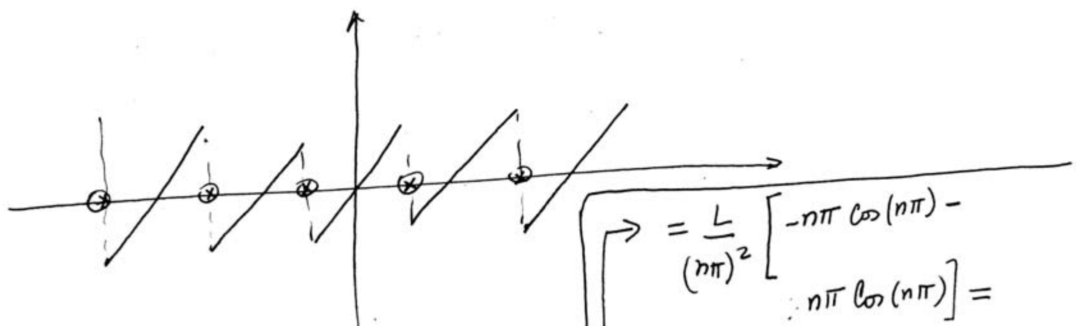
If $f(x^+) = f(x^-)$

the $f(x) = F.S.$

At $x = L$ and $x = -L$, we have to consider the periodic extension first and then compute $f(L^+)$ and $f(L^-)$.

The F.S. will converge to $\dfrac{f(L^+) + f(L^-)}{2}$.

<u>Sketch F.S. of $f(x) = x$</u>



<u>Compute F.S. of $f(x) = x$.</u>

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = \frac{1}{2L}\left.\frac{x^2}{2}\right|_{-L}^{L} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x \cos\frac{n\pi}{L}x}_{\text{odd fn.}}\,dx = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi}{L}x\,dx =$$

$$= \frac{1}{L}\left[\frac{\sin\left(\frac{n\pi}{L}x\right) - \frac{n\pi}{L}x\cos\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^2}\right]_{-L}^{L} =$$

$$= \frac{L}{(n\pi)^2}\left[-n\pi\cos(n\pi) - n\pi\cos(n\pi)\right] =$$

$$= \frac{-L}{(n\pi)^2}\,2n\pi\cos(n\pi)$$

$$= \frac{2L}{n\pi}(-1)(-1)^n = \frac{2L}{n\pi}(-1)^{n+1}$$

Therefore, the Fourier Series of $f(x) = x$ is given by

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{L} x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{L} x\right)$$

$$= \frac{2L}{\pi}\left[+ \sin\left(\frac{\pi}{L} x\right) - \frac{1}{2} \sin\left(\frac{2\pi}{L} x\right) + \frac{1}{3} \sin\left(\frac{3\pi}{L} x\right) + \dots \right]$$

Show MAPLE output!

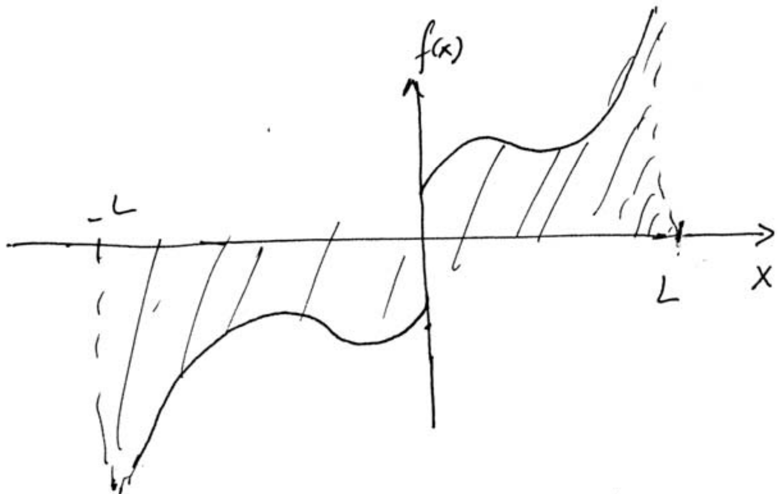
3.3 Fourier Cosine and Sine Series.

In our previous examples the coefficients $a_0 = 0$ and $a_n = 0$, $n = 1, 2, \dots$

Therefore.

F.S. reduced to $\displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$.

This is true in general for odd functions in the interval $[-L, L]$. In fact,



Since if $f(x)$ is odd then,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = 0$$ (see graph above area under curve)

$$a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos \frac{n\pi}{L} x}_{\text{even}}}_{\text{odd}} \, dx \underset{\text{Symm. interval}}{=} 0 .$$

Therefore,

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi}{L} x\right)$$

Also, the coefficients b_n may be simplified

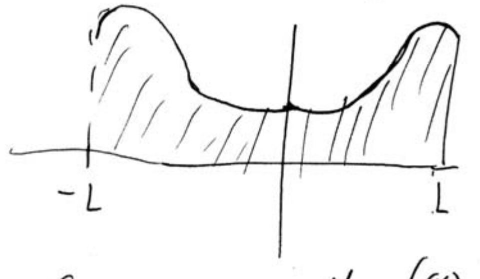
If $g(x)$ is even in the interval $[-L, L]$.

then
$$\int_{-L}^{L} g(x)\,dx = \int_{-L}^{0} g(x)\,dx + \int_{0}^{L} g(x)\,dx$$

Now,
$$\int_{-L}^{0} g(x)\,dx \overset{\substack{y=-x \\ dy=-dx}}{=} \int_{L}^{0} g(-y)(-dy) = -\int_{L}^{0} g(-y)\,dy = -\int_{L}^{0} g(y)\,dy$$
$$= \int_{0}^{L} g(y)\,dy.$$

$$\therefore \quad \int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$$

This can be easily observed from a graph.



Summarizing, if $f(x)$ is an odd function (piecewise smooth) in the interval $[-L, L]$, then its Fourier series reduces to

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where
$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\,dx.$$

FOURIER SINE SERIES.

The Dirichlet IBVP for heat cond.

$$\begin{cases} u_t = k u_{xx}, & 0<x<L, \quad t>0 \\ u(x,0) = f(x), & 0<x<L. \\ u(0,t) = 0, & u(L,t) = 0 \end{cases}$$

has a unique solution given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-ik\left(\frac{n\pi}{L}\right)^2 t}$$

where $$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

and $$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Since $f(x)$ is only defined in $(0,L)$, we can not say is even or odd ~~on~~ $[-L,L]$

Now, we only know F.S. for $f(x)$ defined on $[-L,L]$. and in general they consists of both a series with $\sin\frac{n\pi}{L}x$ and another series with $\cos\left(\frac{n\pi}{L}x\right)$.

In our case, we can extend $f(x)$ on $[0,L]$ to the interval $[-L,L]$ in a way that the new function $\bar{f}(x)$ is odd on $[-L,L]$.

$$\bar{f}(x) = \begin{cases} f(x), & 0<x<L \\ -f(-x), & -L<x<0 \end{cases}$$

$\bar{f}(x)$ is called <u>the odd extension of $f(x)$</u>. If $f(x)$ is piecewise smooth on $(0,L)$, then $\bar{f}(x)$ is also piecewise smooth on $(-L,L)$.

Therefore, $\bar{f}(x)$ F.S. reduces to only sines.

$$\bar{f}(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right), \quad -L<x<L$$

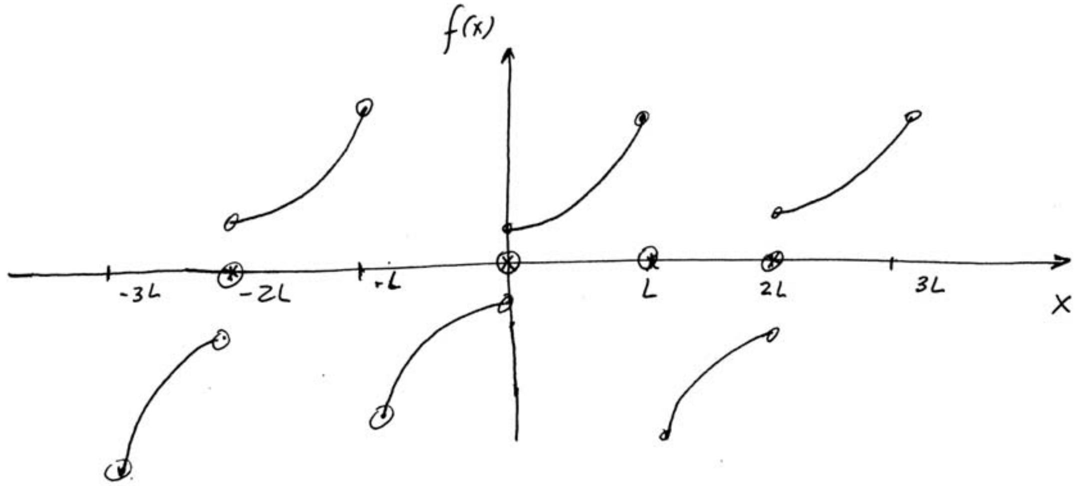
In particular on the interval $[0,L)$.

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)} \qquad (10.1)$$

where $$\boxed{B_n = \frac{2}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right)dx.} \qquad (10.2)$$

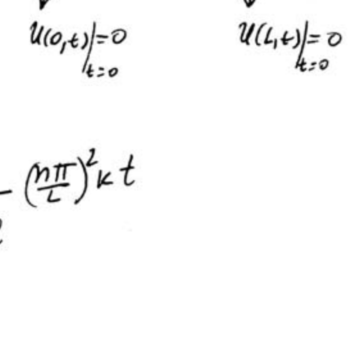
We call (10.1) with the coefficients (10.2) Fourier Sine Series of $f(x)$ on $(0, L)$.

Graph: Sketch the F. sine series of $f(x) = e^x$ on $[0, L]$.



Look to the example in book page 98.

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = 100 = f(x) \end{cases}$$



Clearly discontinuous at the ends

Soln is given by

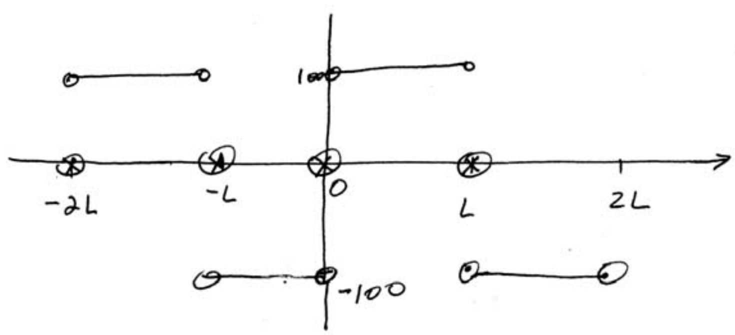
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}$$

And I.C. allows to find $\underline{\underline{B_n\text{'s}}}$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\tfrac{n\pi}{L}x\right) = 100, \qquad 0<x<L$$

where $$B_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi}{L}x\right)dx.$$

Graph of $u(x,0)$ is



Thus, we clearly see that

$u(0,0)=0$, $u(L,0)=0$ and $u(x,0)=100$, $0<x<L$.

The IBVP is completely satisfied by the series soln. (assuming derivation term by term is allowed).